Some incidence theorems and integrable discrete equations

V.E. Adler

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Some incidence theorems of projective geometry admit interpretation as integrable discrete equations. We consider several examples:

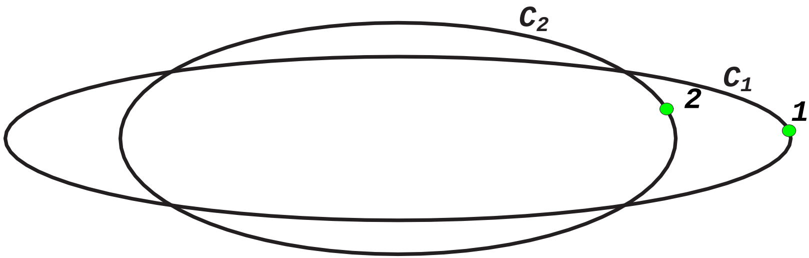
- Yang-Baxter mapping on the linear pencil of conics [1]
- planar quadrilateral lattices [2]
- Möbius generalization of Pascal theorem and discrete Schwarz-BKP equation [2]

[1] V.E. Adler, A.I. Bobenko, Yu.B. Suris, Geometry of Yang-Baxter maps: pencils of conics and quadrirational mappings, math.QA/0307009

[2] V.E. Adler, Some. . . , nlin.SI/0409065

Yang-Baxter mapping on the linear pencil of conics

Let X_1, X_2 be points on the conics C_1, C_2 respectively.

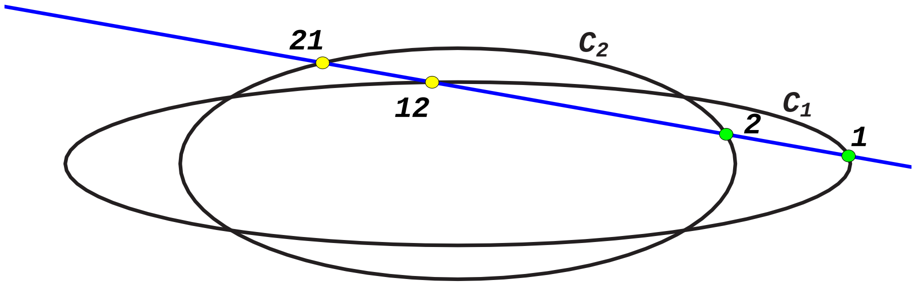


Yang-Baxter mapping on the linear pencil of conics

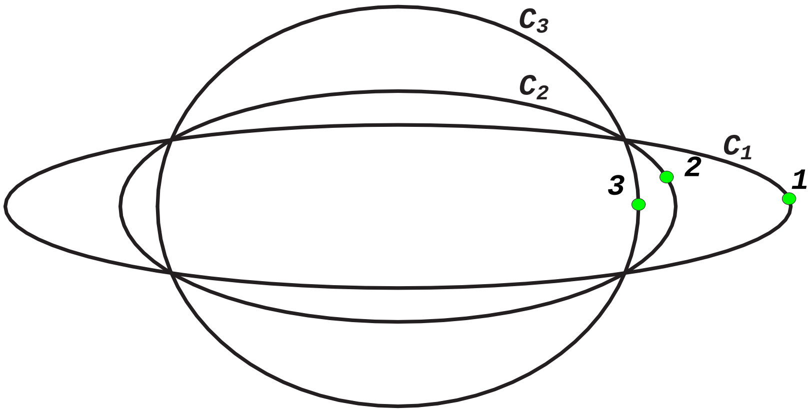
Let X_1, X_2 be points on the conics C_1, C_2 respectively.

The mapping $R_{12} : C_1 \times C_2 \to C_1 \times C_2$ is defined as follows:

$$X_{12} = X_1X_2 \cap C_1, \quad X_{21} = X_1X_2 \cap C_2$$

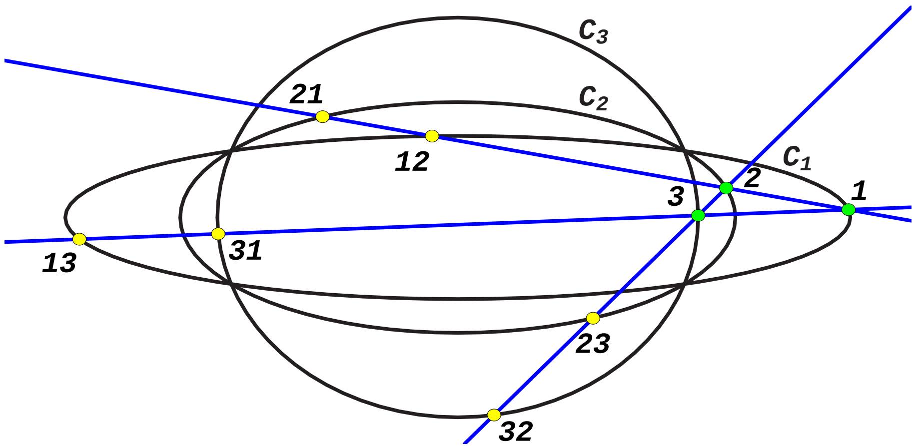


Consider initial data on three conics from the linear pencil.



Consider initial data on three conics from the linear pencil.

Apply the mappings $R_{ij} : (X_i, X_j) \mapsto (X_{ij}, X_{ji})$.

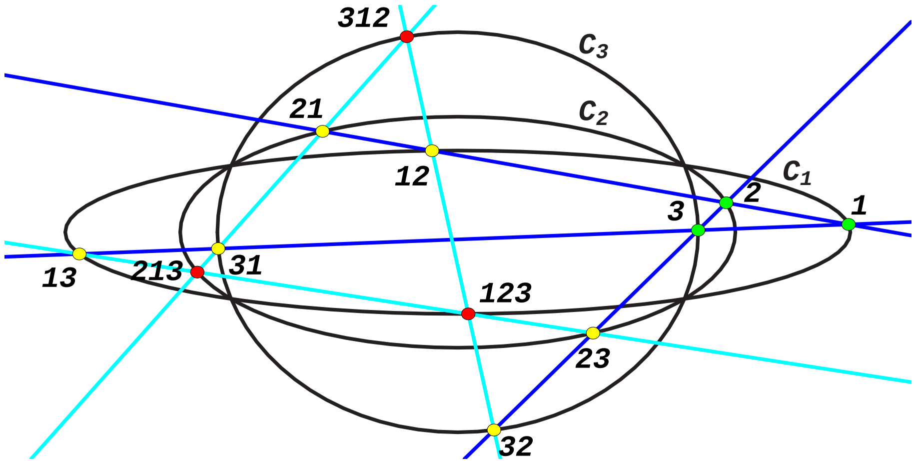


Consider initial data on three conics from the linear pencil.

Apply the mappings $R_{ij} : (X_i, X_j) \mapsto (X_{ij}, X_{ji})$.

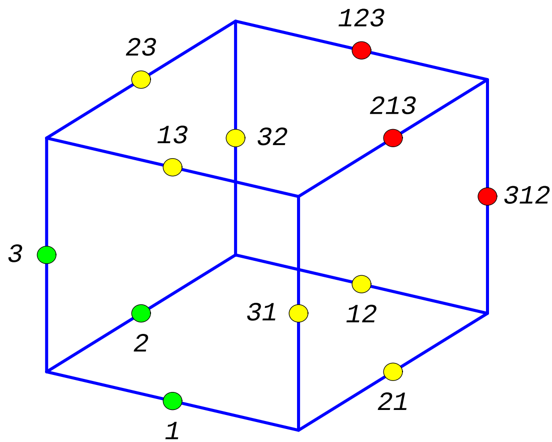
Apply them again. Let $R_{ij} : (X_{ik}, X_{jk}) \mapsto (X_{ikj}, X_{jki})$.

Theorem 1. $X_{ijk} = X_{ikj}$



12 points and 6 straight lines can be identified with the edges and faces of a cube. Parallel edges correspond to the same conic. Parallel faces correspond to the same mapping.

3D-consistency of the mappings R_{ij} means that the two ways of obtaining the point X_{ijk} give the same result.



Under a rational parametrization of conics, R_{12} is a rational mapping on $\mathbb{CP}^1 \times \mathbb{CP}^1$. There are 5 types of linear pencils of conics $C_i = C + \alpha_i K$, leading to the list

$$x_{ij} = \alpha_i x_j \frac{(1-\alpha_2)x_1 + \alpha_2 - \alpha_1 + (\alpha_1 - 1)x_2}{\alpha_2(1-\alpha_1)x_1 + (\alpha_1 - \alpha_2)x_2 x_1 + \alpha_1(\alpha_2 - 1)x_2}$$

$$x_{ij} = \frac{x_j}{\alpha_i} \cdot \frac{\alpha_1 x_1 - \alpha_2 x_2 + \alpha_2 - \alpha_1}{x_1 - x_2}$$

$$x_{ij} = \frac{x_j}{\alpha_i} \cdot \frac{\alpha_1 x_1 - \alpha_2 x_2}{x_1 - x_2}$$

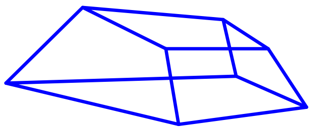
$$x_{ij} = x_j \left(1 + \frac{\alpha_2 - \alpha_1}{x_1 - x_2}\right)$$

$$x_{ij} = x_j + \frac{\alpha_1 - \alpha_2}{x_1 - x_2}$$

The first mapping corresponds to the above pictures with 4-point locus.

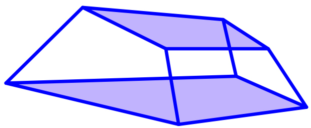
Planar quadrilateral lattices

Consider a combinatorial cube on the plane.



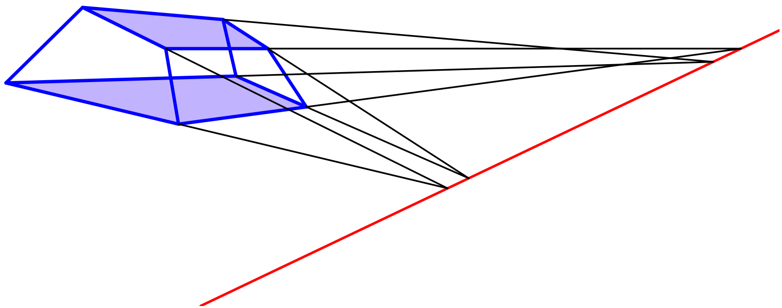
Planar quadrilateral lattices

Consider a combinatorial cube on the plane. If, for some pair of the opposite faces,



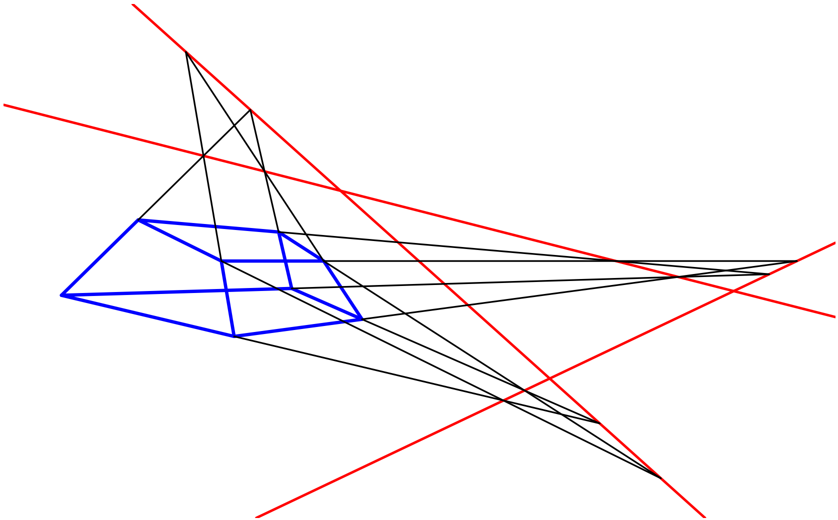
Planar quadrilateral lattices

Consider a combinatorial cube on the plane. If, for some pair of the opposite faces, the intersection points of (the prolongations of) the corresponding edges are collinear,



Planar quadrilateral lattices

Theorem 2. Consider a combinatorial cube on the plane. If, for some pair of the opposite faces, the intersection points of (the prolongations of) the corresponding edges are collinear, then the same is true for any other pair.



Proof. Collinearity of one quadruple of the intersection points allows to construct a combinatorial cube in space, with planar faces, for which our figure is a projection. For such a figure, edges meet on the intersections of 3 pairs of the planes. $\square$

Remark. 8 vertices of the cube + 12 intersection points and 12 sides + 3 lines of intersections form a regular configuration with the symbol $(20_3 15_4)$. This configuration is mentioned in [1], in connection with the following statement (equivalent to Theorem 2):

> Let 3 triangles be perspective with the common center. Then 3 axes of perspective of 3 pairs of triangles meet in one point.

[1] F. Levi, Geometrische Konfigurationen, Leipzig: 1929, p. 143, 202.

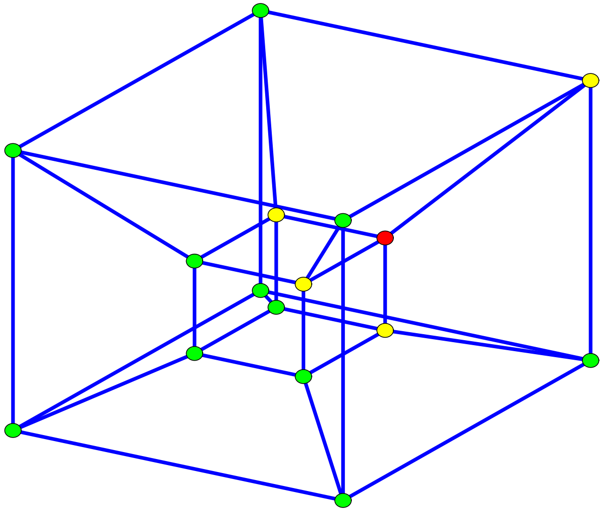
Collinearity of 4 intersection points is a condition, which allows to construct any vertex of the combinatorial cube by the other ones. Let $X, X_1, \ldots, X_{23}$ be given, then X_{123} is defined by

$$\begin{gathered} A = XX_1 \cap X_3X_{13}, \quad B = XX_2 \cap X_3X_{23} \\ A' = X_2X_{12} \cap AB, \quad B' = X_1X_{12} \cap AB \\ X_{123} = A'X_{23} \cap B'X_{13}. \end{gathered} \tag{1}$$

This defines the mapping $R : (\mathbb{CP}^2)^7 \to \mathbb{CP}^2$. Accordingly to the proof of the Theorem 2 and results of [1], this mapping is *4D-consistent*.

[1] A. Doliwa, P.M. Santini, Multidimensional quadrilateral lattices are integrable. *Phys. Lett. A* 233:265–372, 1997.

This means that if we consider a combinatorial hypercube such that each of its 3D faces carries the mapping R, then the point X_{1234} (red) is obtained from the initial data (green) in 4 different ways without contradiction.

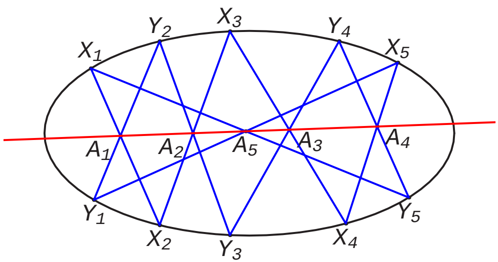


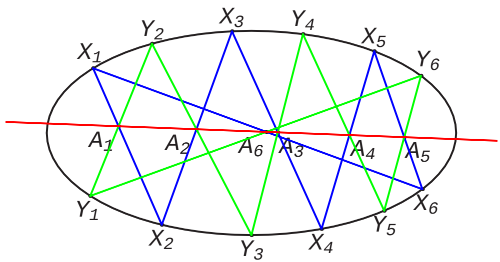
Möbius generalization of Pascal theorem and dSBKP equation

Theorem 3 (Möbius). Let $X_1, Y_1, \ldots, X_N, Y_N$ be points on a conic. Consider the intersection points $A_j = X_jX_{j+1} \cap Y_jY_{j+1}$, $j = 1, \ldots N-1$ and

$$A_N = \begin{cases} X_NY_1 \cap Y_NX_1 & \text{if} \quad N = 2n+1, \\ X_NX_1 \cap Y_NY_1 & \text{if} \quad N = 2n. \end{cases}$$

If all of these points except possibly one are collinear then the same is true for the remaining point.





At $N =$ we have two quadrilaterals $X_1X_2X_3X_4$ and $Y_1Y_2Y_3Y_4$ such that the corresponding sides meet on a straight line. Identifying these quadrilaterals with the opposite faces of a combinatorial cube, we obtain immediately, that the mapping (1) admits the *reduction to a conic section*.

The rational parameters x of the points X on the conic satisfy the discrete Schwarz-BKP, or double cross-ratio equation

$$\frac{(x - x_{12})(x_{13} - x_{23})}{(x_{12} - x_{13})(x_{23} - x)} = \frac{(x_{123} - x_3)(x_2 - x_1)}{(x_3 - x_2)(x_1 - x_{123})}.$$

[1] F.A. Möbius, Verallgemeinerung des Pascal'schen Theorems das in einen Kegelschnit beschriebene Sechseck betreffend. *J. Reine Angew. Math.* 36:216–220, 1848.

[2] B.G. Konopelchenko, W.K. Schief, Reciprocal figures, graphical statics and inversive geometry of the Schwarzian BKP hierarchy. *Stud. Appl. Math.* 109(2):89–124, 2002.